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Almost Projective and Almost Injective Modules

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Abstract—The structure of rings over which every right module is almost injective is clarified. The regular and I-finite rings over which every right module is almost projective are described.

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1. INTRODUCTION

All rings are assumed to be associative and with unit, and the modules are assumed to be unitary. The words like "semi-Artinian ring" mean that the corresponding conditions hold both from the right and from the left.

Let M and N be right R-modules. The module M is said to be *almost N-injective* if, for every submodule N' of N and for every homomorphism $f\colon N' \to M$, either there is a homomorphism $g\colon N \to M$ such that $f = g\iota$ or there is a nonzero idempotent $\pi \in \operatorname{End}_R(N)$ and a homomorphism $h\colon M \to \pi(N)$ for which the equality $hf = \pi\iota$ holds, where $\iota\colon N' \to N$ stands for the natural embedding. The module M is said to be *almost injective* if it is almost injective with respect to every right R-module. The notion of almost projective module is defined in a dual way. A module M is said to be *almost N-projective* if, for every natural homomorphism $g\colon N \to N/K$ and every homomorphism $f\colon M \to N/K$, either there is a homomorphism $h\colon M \to N$ such that $f = gh$ or there are a nonzero direct summand N' of N and a homomorphism $h'\colon N' \to M$ for which $g\iota = fh'$, where $\iota\colon N' \to N$ stands for the natural embedding. A module M is said to be *almost projective* if it is almost projective with respect to every right R-module.

The notions of almost injective module and almost projective module were first studied in the papers [1]–[7] by Harada and his colleagues and disciples. Note that, in [7], an almost projective right R-module was defined as a module which is almost projective with respect to every finitely generated right R-module. Recently, almost injective modules were considered in [8]–[12]. In [10], the problem of describing rings over which every module is almost injective was posed. In some special cases, this problem was solved in [10] (in particular, in the case of semiperfect rings). In the present paper, the structure of rings over which every module is almost injective is clarified in the general case. We also obtain characterizations of modules M for which every simple module in the category $\sigma(M)$ is almost projective (almost injective, respectively). We describe regular and I-finite rings over which every right module is almost projective.

Let M and N be right R-modules. Denote by $\sigma(M)$ the full subcategory of all right R-modules consisting of all M-subgenerated modules. If $N \in \sigma(M)$, then denote the injective hull of the module N in the category $\sigma(M)$ by $E_M(N)$. By $J(M)$, denote the Jacobson radical of the module M.

By the *Loewy series* of a module M we mean the ascending chain

$$0 \subset \operatorname{Soc}_1(M) = \operatorname{Soc}(M) \subset \cdots \subset \operatorname{Soc}_\alpha(M) \subset \operatorname{Soc}_{\alpha+1}(M) \subset \cdots,$$

where

$$\operatorname{Soc}_\alpha(M)/\operatorname{Soc}_{\alpha-1}(M) = \operatorname{Soc}(M/\operatorname{Soc}_{\alpha-1}(M))$$

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